Extendibility of quantum Gaussian states

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Joint work with Ludovico Lami (Nottingham), Gerardo Adesso (Nottingham)
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QuILT Day

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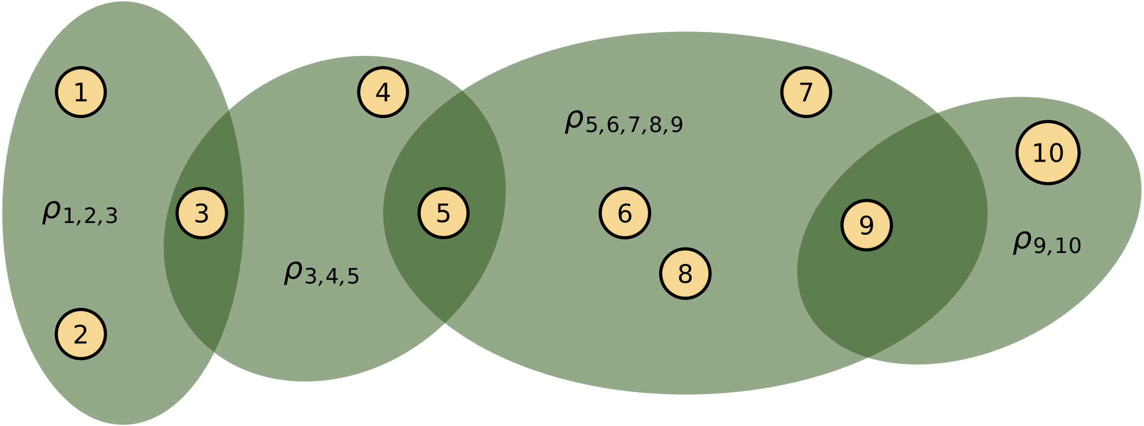


Does there exist a global state $\tilde{\rho}_{1,2,\ldots,10}$ with the given marginals?

This is the *quantum marginal problem*.

REVIEWS OF MODERN PHYSICS VOLUME 35, NUMBER 3 JULY 1963

Structure of Fermion Density Matrices

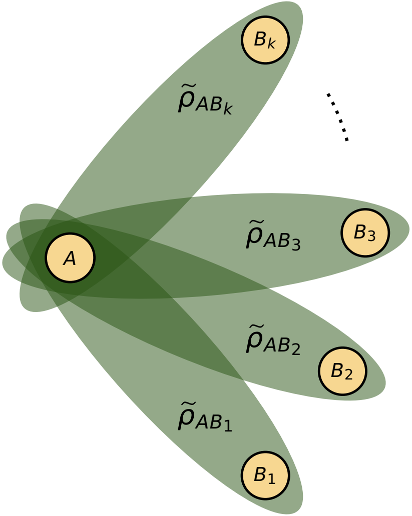
A. J. Coleman
Queen's University, Kingston, Ontario, Canada

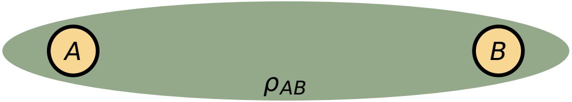
1. INTRODUCTION

CAN the wave function be eliminated from quantum mechanics and its role be taken over, in the discussion of physical systems, by reduced density matrices? The author has believed in the affirmative answer to this question for over ten years. In the present paper, he attempts to muster the main current evidence in support of this belief. Prior to the Hylleraas Symposium, the available evidence, probably, would not have convinced the average physi- terest in the density matrix approach to the N-body problem stated, "It has frequently been pointed out that a conventional many-electron wave function tells us more than we need to know. . . . There is an instinctive feeling that matters such as electron correlation should show up in the two-particle density matrix . . . but we still do not know the conditions that must be satisfied by the density matrix. Until these conditions have been elucidated, it is going to be very difficult to make much progress along these lines."

and with which the present paper is concerned. We refer to the problem as the *N-representability problem*—how can we recognize that an alleged two-particle density matrix is, in fact, the reduced density matrix of a system of N-indistinguishable particles. Apart from occasional side remarks, we restrict our attention to the case of fermions.

The *N-representability problem* in quantum chemistry.
(QMA-complete [Liu06].)





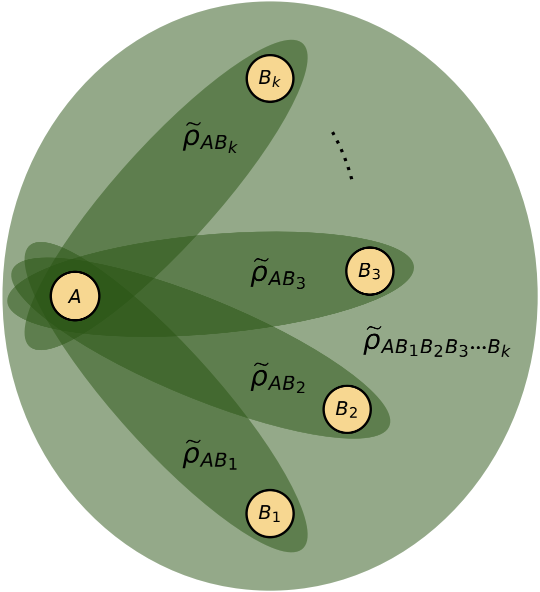
$$\tilde{\rho}_{AB_1} = \rho_{AB}$$

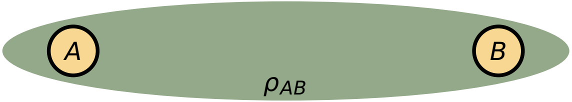
$$\tilde{\rho}_{AB_2} = \rho_{AB}$$

$$\tilde{\rho}_{AB_3} = \rho_{AB}$$

$$\vdots$$

$$\tilde{\rho}_{AB_k} = \rho_{AB}$$





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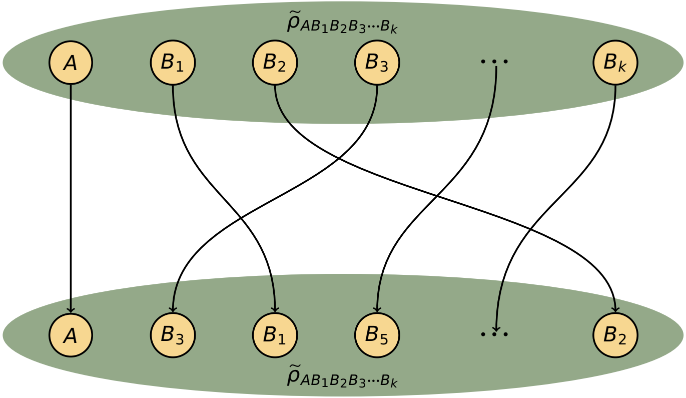
$$\vdots$$

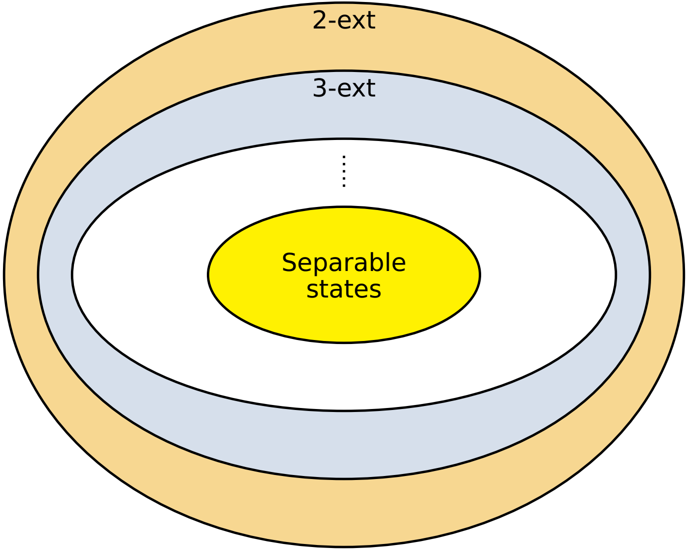
$$\widetilde{\rho}_{AB_k} = \rho_{AB}$$

$\widetilde{\rho}_{AB_1B_2B_3\cdots B_k}$ is a *k-extension* of ρ_{AB}.

Alternate definition: $\widetilde{\rho}_{AB_1B_2B_3\cdots B_k}$ invariant under any permutation of $B_1, \ldots, B_k$.



$$\mathrm{Tr}_{B_2B_3\cdots B_k}\left[\widetilde{\rho}_{AB_1B_2B_3\cdots B_k}\right] = \rho_{AB}.$$



(See [DPS02,04]; [BPS17].)

When is a quantum state k-extendible?

Given state ρ_{AB} and $k \geq 2$:

$$\begin{array}{lll} \text{find} & \widetilde{\rho}_{AB_1\cdots B_k} \geq 0 & \\ \text{subject to} & \widetilde{\rho}_{AB_i} = \rho_{AB} & \forall\, i \in \{1, 2, \ldots, k\}. \end{array}$$

This is a semi-definite program (SDP).

- For $k = 2$ and ρ_{AB} any two-qubit state [CJKLZ14]:

$$\mathrm{Tr}[\rho_{AB}^2] - \mathrm{Tr}[\rho_B^2] \geq 4\sqrt{\det(\rho_{AB})}$$

  is necessary and sufficient. In higher dimensions and higher k, only partial results known. (See, e.g., [JV13])

★ What about infinite-dimensional systems (e.g., quantum optical systems)?

Our work

- Study k-extendibility of states ρ_{AB} of continuous-variable systems.
- Obtain analytic necessary condition for all states for all k.
- For Gaussian states:
  - Obtain simple finite-dimensional SDP.
  - For $n_B = 1$ mode, obtain an analytic necessary and sufficient condition for all k.
- Bounds on the distance of k-extendible Gaussian states from the set of separable states.
- Bounds on the entanglement of k-extendible Gaussian states.

Continuous-variable systems

For a system of n modes:

- Position and momentum operators $x_1, \ldots, x_n$ and $p_1, \ldots, p_n$. Let $\vec{r} = (x_1, p_1, \ldots, x_n, p_n)^\intercal$.
- Mean vector $\vec{s}_\rho$ and covariance matrix V_ρ of a state ρ:

$$(\vec{s}_\rho)_k = \mathrm{Tr}[r_k \rho], \quad (V_\rho)_{k,\ell} = \mathrm{Tr}[(\bar{r}_k \bar{r}_\ell + \bar{r}_\ell \bar{r}_k)\rho] \quad \forall\ 1 \le k, \ell \le 2n,$$

  $\bar{r}_k = r_k - (\vec{s}_\rho)_k$, etc.
- For any quantum state ρ:

$$V_\rho + \mathrm{i}\Omega_n \ge 0, \quad \Omega_n \coloneqq \mathbb{1}_n \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \bigoplus_{j=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Gaussian states

- Have a Gaussian characteristic function.
- Fully described by mean and covariance matrix.
- Faithful Gaussian state on n modes can be written as

$$\rho^{\mathrm{G}} = \frac{\mathrm{e}^{-\beta H}}{\mathrm{Tr}[\mathrm{e}^{-\beta H}]}, \quad \beta > 0, \quad H = r^{\mathsf{T}} A r + r^{\mathsf{T}} s,$$

  where $s \in \mathbb{R}^{2n}$ and A is $2n \times 2n$ real symmetric.
- Example: single-mode thermal state:

$$\theta(\bar{n}) = \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1} \right)^n |n\rangle\langle n|.$$

k-extendibility of CV states

Consider bipartite quantum state ρ_{AB} with $n_A + n_B$ modes and

$$\vec{s}_{AB} \equiv \vec{s}_{\rho_{AB}} = \begin{pmatrix} \vec{s}_A \\ \vec{s}_B \end{pmatrix}, \quad V_{AB} \equiv V_{\rho_{AB}} = \begin{pmatrix} V_A & X \\ X^\intercal & V_B \end{pmatrix}.$$

★ **First reduction**

Restrict to zero-mean states. k-extendibility invariant under local unitaries.

$$\rho'_{AB} = \left(D(-\vec{s}_A) \otimes D(-\vec{s}_B)\right) \rho_{AB} \left(D(-\vec{s}_A) \otimes D(-\vec{s}_B)\right)^\dagger \quad \text{has zero mean.}$$

ρ_{AB} k-extendible if and only if ρ'_{AB} k-extendible.

k-extendibility of CV states

★ **Second reduction**

If ρ_{AB} k-extendible, then covariance matrix of a k-extension has the form

$$\widetilde{V}_{AB_1\cdots B_k}(Y) = \begin{pmatrix} V_A & X & X & \cdots & X \\ X^\intercal & V_B & Y & \cdots & Y \\ X^\intercal & Y & V_B & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & Y \\ X^\intercal & Y & \cdots & Y & V_B \end{pmatrix}$$

for some Y such that $\widetilde{V}_{AB_1\cdots B_k}(Y) + \mathrm{i}\Omega \geq 0$.

k-extendibility of CV states

★ **Third reduction (and first result)**
For Gaussian states, k-extendible if and only if k-extension is Gaussian.
$\Rightarrow$ Gaussian state ρ^{G}_{AB} k-extendible *if and only if*

$$\begin{array}{ll} \text{exists} & Y \\ \text{such that} & \widetilde{V}_{AB_1\cdots B_k}(Y) + i\Omega \geq 0 \end{array}$$

This is an SDP for k-extendibility of Gaussian states.

$$\widetilde{V}_{AB_1\cdots B_k}(Y) = \begin{pmatrix} V_A & X & X & \cdots & X \\ X^{\mathsf{T}} & V_B & Y & \cdots & Y \\ X^{\mathsf{T}} & Y & V_B & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & Y \\ X^{\mathsf{T}} & Y & \cdots & Y & V_B \end{pmatrix}$$

Main result

- **All states:** if ρ_{AB} k-extendible, then there exists Δ_B such that

$$\Delta_B \geq \mathrm{i}\Omega_B \quad \text{and} \quad V_{AB} \geq \mathrm{i}\Omega_A \oplus \left(\left(1 - \frac{1}{k}\right) \Delta_B + \frac{1}{k} \mathrm{i}\Omega_B \right).$$

- **Gaussian states:** ρ^{G}_{AB} k-extendible *if and only if* there exists Δ_B such that

$$\Delta_B \geq \mathrm{i}\Omega_B \quad \text{and} \quad V_{AB} \geq \mathrm{i}\Omega_A \oplus \left(\left(1 - \frac{1}{k}\right) \Delta_B + \frac{1}{k} \mathrm{i}\Omega_B \right).$$

  If $n_B = 1$, then ρ^{G}_{AB} k-extendible *if and only if*

$$V_{AB} \geq \mathrm{i}\Omega_A \oplus \left(-\left(1 - \frac{2}{k}\right) \mathrm{i}\Omega_B \right).$$

Proof sketch

★ Main tool: the *Schur complement*:

$$M = \begin{pmatrix} P & Z \\ Z^\dagger & Q \end{pmatrix} \geq 0 \quad \Leftrightarrow \quad \underbrace{Q - Z^\dagger P^{-1} Z}_{\text{Schur complement of M wrt P}} \geq 0.$$

Proof sketch

★ Main tool: the *Schur complement*:

$$M = \begin{pmatrix} P & Z \\ Z^\dagger & Q \end{pmatrix} \geq 0 \quad \Leftrightarrow \quad \underbrace{Q - Z^\dagger P^{-1} Z}_{\text{Schur complement of M wrt P}} \geq 0.$$

Suppose ρ_{AB} is k-extendible. Then there exists Y such that $\widetilde{V}_{AB_1\cdots B_k}(Y) + i\Omega_{AB_1\cdots B_k} \geq 0$.

$$\widetilde{V}_{AB_1\cdots B_k}(Y) - i\Omega_{AB_1\cdots B_k} = \begin{pmatrix} V_A - i\Omega_A & X & X & \cdots & X \\ X^T & V_B - i\Omega_B & Y & \cdots & Y \\ X^T & Y & V_B - i\Omega_B & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & Y \\ X^T & Y & \cdots & Y & V_B - i\Omega_B \end{pmatrix}$$

Proof sketch

Let $\Delta_B \coloneqq V_B - Y$. Then we get

$$\mathrm{i}\Omega_B \le \Delta_B \le \frac{k}{k-1}\left(V_B - X^\mathsf{T}(V_A - \mathrm{i}\Omega_A)^{-1}X\right) - \frac{1}{k-1}\mathrm{i}\Omega_B. \qquad (*)$$

Apply Schur complement to right-hand side to get

$$V_{AB} \ge \mathrm{i}\Omega_A \oplus \left(\left(1 - \frac{1}{k}\right)\Delta_B + \frac{1}{k}\mathrm{i}\Omega_B\right).$$

For $n_B = 1$ $(*)$ consists of only 2×2 matrices.

Proof sketch

For $n_B = 1$, all matrices in $(*)$ are 2×2:

$$\mathrm{i}\Omega_B \le \Delta_B \le \frac{k}{k-1}\left(V_B - X^\top (V_A - \mathrm{i}\Omega_A)^{-1} X\right) - \frac{1}{k-1}\mathrm{i}\Omega_B.$$

★ For M, N 2×2 Hermitian, there exists real symmetric matrix R satisfying $M \le R \le N$ if and only if $M \le N$ and $\overline{M} \le N$.

Second condition leads to $V_B - X^\top (V_A - \mathrm{i}\Omega_A)^{-1} X \ge -\left(1 - \frac{2}{k}\right)\mathrm{i}\Omega_B$.

Apply Schur complement to get

$$V_{AB} \ge \mathrm{i}\Omega_A \oplus \left(-\left(1 - \frac{2}{k}\right)\mathrm{i}\Omega_B\right).$$

Entanglement of Gaussian states

If ρ_{AB} k-extendible, then there exists Δ_B such that

$$\Delta_B \geq \mathrm{i}\Omega_B \quad \text{and} \quad V_{AB} \geq \mathrm{i}\Omega_A \oplus \left(\left(1 - \frac{1}{k}\right)\Delta_B + \frac{1}{k}\mathrm{i}\Omega_B\right).$$

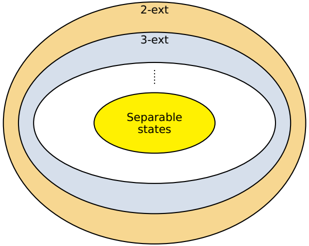
Take limit $k \to \infty$:

$$V_{AB} \geq \mathrm{i}\Omega_{AB} \oplus \Delta_B.$$

ρ_{AB} separable if and only if there exists Δ_B such that

$$\Delta_B \geq \mathrm{i}\Omega_B \quad \text{and} \quad V_{AB} \geq \mathrm{i}\Omega_A \oplus \Delta_B.$$

Reproduces known result from [LSA18].



Entanglement of Gaussian states

Distance to the set of separable states:

$$\left\|\rho_{AB} - \mathrm{SEP}(A{:}B)\right\|_1 = \inf_{\sigma_{AB}\in\mathrm{SEP}(A:B)} \|\rho_{AB} - \sigma_{AB}\|_1 ,$$

$$E_R^{\alpha}(\rho_{AB}) = \inf_{\sigma_{AB}\in\mathrm{SEP}(A:B)} D_\alpha(\rho_{AB}\|\sigma_{AB}), \quad \alpha > 0.$$

★ How far is a Gaussian k-extendible state ρ_{AB}^{G} from the set of separable states?

$$\left\|\rho_{AB}^{\mathrm{G}} - \mathrm{SEP}(A{:}B)\right\|_1 \leq \frac{2(n_A + n_B)}{k},$$

$$E_R^{\alpha}(\rho_{AB}^{\mathrm{G}}) \leq (n_A + n_B)\log\left(\frac{k}{k-1}\right), \quad \alpha > 0.$$

Summary and outlook

- Necessary and sufficient condition for k-extendibility of all Gaussian states as a finite-dimensional SDP.
- Full characterization for $n_B = 1$.
- Application: bounds on the distance to the set of separable states.

Directions for future work

- Extend full characterization to $n_B > 1$.
- Bound non-asymptotic quantum capacity of single-mode Gaussian channels.

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